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Your Roll No.....

Sr. No. of Question Paper : 1381 C

Unique Paper Code : 32351301

Name of the Paper : BMATH 305 – Theory of
Real Functions

Name of the Course : CBCS (LOCF) B.Sc. (H)
Mathematics

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

Deshbandhu College Library
Kalkaji, New Delhi-110019

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any two parts from each question.
3. All questions are compulsory.

1. (a) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A and $f: A \rightarrow \mathbb{R}$, then define limit of function f at c .

Use $\varepsilon - \delta$ definition to show that $\lim_{x \rightarrow 1} \frac{x}{x+1} = \frac{1}{2}$.

(6)

P.T.O.

(b) Let $f: A \rightarrow \mathbb{R}$, $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A . Then show that $\lim_{x \rightarrow c} f(x) = L$ if and only if for every sequence $\langle x_n \rangle$ in A that converges to c such that $x_n \neq c$, $\forall n \in \mathbb{N}$, the sequence $\langle f(x_n) \rangle$ converges to L . (6)

(c) Show that $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x^2}\right)$ does not exist in \mathbb{R} but $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x^2}\right) = 0$. (6)

2. (a) Let $A \subseteq \mathbb{R}$, $f: A \rightarrow \mathbb{R}$, $g: A \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A . Show that if f is bounded on a neighborhood of c and $\lim_{x \rightarrow c} g(x) = 0$, then $\lim_{x \rightarrow c} (fg)(x) = 0$. (6)

(b) Let $f(x) = e^{1/x}$ for $x \neq 0$, then find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0^+} f(x)$. (6)

(c) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational} \end{cases}$$

Find all the points at which f is continuous.

(6)

3. (a) Let $A \subseteq \mathbb{R}$ and let f and g be real valued functions on A . Show that if f and g are continuous on A then their product fg is continuous on A . Also, give examples of two functions f and g such that both are discontinuous at a point $c \in A$ but their product is continuous at c . (7½)
- (b) State and prove Boundedness Theorem for continuous functions on a closed and bounded interval. (7½)
- (c) State Maximum-Minimum Theorem. Let $I = [a, b]$ and $f: I \rightarrow \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I . Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all x in I . (7½)
4. (a) Let $A \subseteq \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in A$. Show that if f is continuous at $c \in A$, then \sqrt{f} is continuous at c . (6)
- (b) Show that every uniformly continuous function on $A \subseteq \mathbb{R}$ is continuous on A . Is the converse true? Justify your answer. (6)
- (c) Show that the function $f(x) = \frac{1}{x^2}$, $x \neq 0$ is uniformly continuous on $[a, \infty)$, for $a > 0$ but not uniformly continuous on $(0, \infty)$. (6)

5. (a) Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be functions that are differentiable at c . Prove that if $g(c) \neq 0$, the function f/g is differentiable at c , and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}. \quad (6)$$

- (b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = |x| + |x + 1|$, $x \in \mathbb{R}$. Is f differentiable everywhere in \mathbb{R} ? Find the derivative of f at the points where it is differentiable. (6)

- (c) State Mean Value Theorem. If $f: [a, b] \rightarrow \mathbb{R}$ satisfies the conditions of Mean Value Theorem and $f'(x) = 0$ for all $x \in (a, b)$. Then prove that f is constant on $[a, b]$. (6)

6. (a) Let I be an open interval and let $f: I \rightarrow \mathbb{R}$ have a second derivative on I . Then show that f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$. (6)

- (b) Find the points of relative extrema of the functions $f(x) = |x^2 - 1|$, for $-4 \leq x \leq 4$. (6)

- (c) Use Taylor's Theorem with $n = 2$ to approximate $\sqrt[3]{1+x}$, $x > -1$. (6)